

Spin relaxation in quantum Hall systems

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We study the spin relaxation in an interacting two-dimensional electron gas in a strong magnetic field for the case that the electron density is close to filling just one Landau sub-level of one spin projection, i.e., for filling factor $\nu \simeq 1$. Assuming the relaxation to be caused by scattering with phonons, we derive the kinetic equations for the electron's spin-density which replace the Bloch equations in our case. These equations are non-linear and their solution depends crucially on the filling factor ν and on the temperature T of the phonon bath. In the limit of $T = 0$ and $\nu = 1$, the solution relaxes asymptotically with a power law inversely proportional to time, instead of following the conventional exponential behavior.

Experiments in two-dimensional (2-d) electronic structures in a strong magnetic field reveal a rich variety of features. Recently, many experiments have focussed on properties related to the electrons' spins [1–8]. Optical measurements of the magnetization and transport measurements of the activation energy are interpreted with the concept of Skyrmion-quasiparticles [9,10]. These can be described as extended spin-textures containing a number of spins which are flipped with respect to the preferential direction set by the magnetic field. Their number is determined by the competition between Zeeman energy and Coulomb interaction of the electrons. While the equilibrium of systems containing Skyrmions has been studied in various ways – both experimentally and theoretically, the Skyrmion's dynamical properties are much less understood. There are indirect measurements of electron spin resonance which use the magnetoresistivity to determine the spin-splitting of the Landau levels [11–13] and which also yield a line width [12].

The first and fundamental question about the dynamics regards the formation of the Skyrmion. A Skyrmion anti-Skyrmion pair has only half the energy of a particle-hole excitation, a spin exciton [9,10]; nevertheless, it needs for its formation a mechanism which flips the electron's spins, since a Skyrmion may contain a large number of overturned spins. The mechanism must be provided from outside the bare electron system of Coulomb interaction and Zeeman energy, because the spin parallel to the magnetic field is a constant of motion. In this work, we wish to analyze the spin relaxation, originating from such a mechanism, specifically for a system of 2-d interacting electrons in a strong magnetic field. As it will be shown below, the result is intriguing and non-trivial for this system, already in the case of a uniform spin-density, and quite different from the well-known Bloch equations [14]. Thus, we will demonstrate our new approach studying the uniform case; this has to be seen as the first step towards a solution of the more general problem of the formation of the Skyrmion.

We consider in the following interacting electrons in two dimensions moving in a strong magnetic field under the

following conditions: the electron density should be close to that of one filled spin-split Landau sub-level, i.e. the filling factor $\nu \simeq 1$, and the Zeeman energy should be much less than the Coulomb energy. Both conditions are relevant for the Skyrmion case. The latter is realized in the experiment by a variation of the g -factor with pressure [6]. In three-dimensional (3-d) semiconductors, various mechanisms of spin relaxation have been studied experimentally (e.g., in optical measurements) and theoretically. Among them are scattering by holes due to the exchange energy, and scattering by phonons and impurities which becomes possible because of the spin-orbit interaction; for a comparison of these different mechanisms, see Ref. [15]. In a system in strong magnetic field, the Zeeman energy has to be taken up by the scatterer in an elementary spin flip. Therefore, we study here a general model of inelastic scattering by phonons. In deriving the kinetic equation which governs the relaxation of the spin-density due to the scattering by phonons, the following three assumptions are made: (i) the 2-d electrons are described by a non-equilibrium distribution which yields *uniform* expectation values for density and spin-density. (ii) the phonons are in thermal equilibrium unperturbed by the electrons. (iii) the electron-phonon coupling is so weak that it can be treated in perturbation theory. The appropriate technique for performing perturbation theory in non-equilibrium systems was introduced by Keldysh [16,17]. It is shown in Ref. [17] how to derive the kinetic equation in the quasi-classical regime. The condition for the applicability of the quasi-classical approximation in the present context of the spin relaxation is given below.

How can phonons flip the spin ? Consider the standard expression of the spin-orbit energy,

$$V_{so} = \frac{1}{2} \boldsymbol{\sigma} \cdot \frac{-e\hbar}{2m^2c^2} \mathbf{E} \times \mathbf{p}, \quad (1)$$

where $\frac{1}{2}\boldsymbol{\sigma}$ and \mathbf{p} are spin and momentum of the electron. The electric field \mathbf{E} has its origin in a piezoelectric distortion of the lattice and it is linear in phonon operators. Thus, the electrons are scattered and their spin is

flipped by the phonons of the 3-d lattice which contains the 2-d electron system at $z = 0$. The orbital part of the electron's single particle states is confined to the lowest Landau level. We denote the projected spin-density of the electrons at (2-d) wave vector \mathbf{q} with $\hat{\mathbf{S}}(\mathbf{q})$ [18]. Then, we get for the electron-phonon part of the Hamiltonian

$$H_{e-ph} = \sum_{\mathbf{Q},s} \hat{\mathbf{S}}(\mathbf{q}) \left(\lambda_s(-\mathbf{Q}) b_{-\mathbf{Q},s} + \lambda_s(\mathbf{Q})^* b_{\mathbf{Q},s}^\dagger \right). \quad (2)$$

$b_{\mathbf{Q},s}^\dagger$ creates a phonon with (3-d) wave vector $\mathbf{Q} = (\mathbf{q}, Q^z)$ and polarization s . The coupling constant $\lambda_s(\mathbf{Q})$ can be derived from (1). It contains the common electronic wave function in the z -direction. With a relation between electric field and phonons as in the model of Refs. [19,20], $\lambda_s(\mathbf{Q})$ is quadratic in momentum. The considerations below are valid for arbitrary $\lambda_s(\mathbf{Q})$.

Having specified the model, we will now study first the kinetic equation for a strongly simplified case in which only two non-interacting electronic states are kept. In a second step, we will then come back to the full model of interacting electrons in two Landau sub-levels coupled to a phonon bath.

Two non-interacting states — The Hamiltonian is given by

$$H_{toy} = -\Delta(c_{\uparrow}^\dagger c_{\uparrow} - c_{\downarrow}^\dagger c_{\downarrow}) + \sum_{\mathbf{q}} \omega_{\mathbf{q}} \left(b_{\mathbf{q}}^\dagger b_{\mathbf{q}} + \frac{1}{2} \right) + \sum_{\mathbf{q}} \lambda_{\mathbf{q}} \left(c_{\downarrow}^\dagger c_{\uparrow} b_{\mathbf{q}} + c_{\uparrow}^\dagger c_{\downarrow} b_{\mathbf{q}}^\dagger \right), \quad (3)$$

where $c_{\uparrow,\downarrow}^\dagger$ ($b_{\mathbf{q}}^\dagger$) are electron (phonon) creation operators. Here, scattering from the \uparrow state to the \downarrow state occurs under absorption of a phonon with an arbitrary wavevector \mathbf{q} , and the same is true for the reverse process. The kinetic equation for the time-dependent average occupation numbers $n_{\uparrow,\downarrow}(t)$ can be derived with the aid of Fermi's golden rule (here, time arguments are omitted):

$$\partial_t n_{\uparrow} = -\partial_t n_{\downarrow} = \frac{1}{\tau} [(1 - n_{\uparrow})n_{\downarrow}(1 + N) - n_{\uparrow}(1 - n_{\downarrow})N]. \quad (4)$$

Eq. (4) is valid in the quasi-stationary limit $\Delta \gg 1/\tau$, and the relaxation time τ is given by

$$\frac{1}{\tau} = \sum_{\mathbf{q}} \lambda_{\mathbf{q}}^2 2\pi \delta(2\Delta - \omega_{\mathbf{q}}). \quad (5)$$

$N = [e^{\beta 2\Delta} - 1]^{-1}$ is the occupation number of the only phonon ($\omega_{\mathbf{q}} = 2\Delta$) which can be effective under energy conservation. The density $\nu = n_{\uparrow}(t) + n_{\downarrow}(t)$ is fixed; thus it is clear that Eq. (4) has two stationary solutions for n_{\uparrow} . Only one of these is physical, the other is unphysical ($n_{\uparrow} > 1$). The kinetic equation (4) is best discussed

in terms of the depolarization, the deviation from the physical solution, $\delta(t) = n_{\uparrow}(\infty) - n_{\uparrow}(t)$:

$$\tau \partial_t \delta(t) = -\delta(t) [\epsilon + \delta(t)]. \quad (6)$$

The crucial parameter ϵ is the difference between the two stationary solutions, $\epsilon = [4N(1 + N) + (\nu - 1)^2]^{1/2}$. It depends strongly on the temperature and the total density. For zero temperature $1/\beta = 0$, $\epsilon = |\nu - 1|$. For $\nu = 1$, on the other hand, $\epsilon \sim 2e^{-\beta\Delta}$ at low temperatures. Now it is obvious that, and under which conditions, linearization in the kinetic equation (6) can fail: Eq. (6) shows two regimes depending on the relative size of ϵ and $\delta(t)$. If $\delta(t)$ is smaller than ϵ , then one can linearize Eq. (6) in $\delta(t)$, and $\delta(t)$ decays exponentially. If $\delta(t)$ is larger than ϵ , then Eq. (6) becomes quadratic in $\delta(t)$, and $\delta(t)$ decays with $1/t$ until it becomes so small that it crosses over into the first regime. This behavior is pictured in Fig. 1 for $\delta(t=0) = 0.25$. The solid line shows the depolarization for $\epsilon = 0.05$. Initially, the curve follows that of the quadratic equation ($\epsilon = 0$, dashed line), until it crosses over to exponential behavior. This limiting exponential behavior is indicated as a dotted line. We conclude that even the solution of the oversimplified model H_{toy} shows a non-trivial behavior, far from being exponential *per se*. It displays a strong dependence on temperature and density. The asymptotic exponential decay of $\delta(t)$ can become arbitrarily slow ($\epsilon \rightarrow 0$) as the temperature approaches 0 and $\nu \rightarrow 1$; and in the limit, $\delta(t) \propto 1/t$.

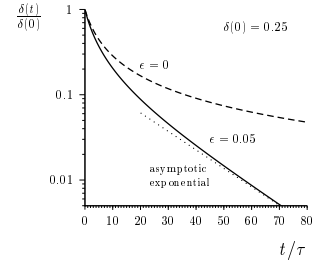


Fig. 1: Depolarization $\delta(t)$ as a function of t/τ for an initial value of $\delta(0) = 0.25$; solid line: $\epsilon = 0.05$, dashed line: $\epsilon = 0$.

Electron-electron interaction — We now turn back to the case of interacting electrons in the two lowest Landau sub-levels split by the Zeeman energy. The electrons are scattered by phonons, see H_{e-ph} , Eq. (2). The electronic part H_e of the total Hamiltonian $H_e + H_{ph} + H_{e-ph}$ is then

$$H_e = -\Delta \sum_p (c_{\uparrow p}^\dagger c_{\uparrow p} - c_{\downarrow p}^\dagger c_{\downarrow p}) + H_{Coul}. \quad (7)$$

Here, $c_{\uparrow p}^\dagger$ ($c_{\downarrow p}^\dagger$) creates a Landau state with linear momentum p and spin \uparrow (\downarrow). The most important ingredient, which was missing in H_{toy} , is the Coulomb interaction H_{Coul} between the electrons, cf. [21]. As stated in

assumption (i) above, the expectation values of density and spin-density of the electrons are presumed to be uniform. The density is conserved and its value is given by the filling factor $\nu \sim 1$. We wish to study the uniform spin-density, $\mathbf{S}(t) = \langle \hat{\mathbf{S}}(\mathbf{q} = 0) \rangle$,

$$\mathbf{S}(t) = \frac{1}{N_\Phi} \sum_p \langle (c_{\uparrow p}^\dagger, c_{\downarrow p}^\dagger)(t) \frac{1}{2} \boldsymbol{\sigma} \begin{pmatrix} c_{\uparrow p} \\ c_{\downarrow p} \end{pmatrix}(t) \rangle. \quad (8)$$

N_Φ denotes the number of states in a Landau sub-level; if all \uparrow states are populated and all \downarrow states are empty, $\mathbf{S}(t) = \frac{1}{2} \hat{e}_z$. The derivation of the kinetic equation for $\mathbf{S}(t)$ can be performed with the method of Keldysh [16]. The kinetic equation contains a precession term, which is already present without any phonon scattering, and the main term resulting from the collisions with the phonons. In leading (second) order perturbation theory in the electron-phonon coupling, the collision integral becomes

$$\begin{aligned} \partial_t S^m(t)|_{coll} = & -i \int_{-\infty}^t dt' \sum_{\mathbf{Q}} \sum_{j,k,l} \epsilon^{mjkl} \\ & [\mathcal{C}^{kl}(\mathbf{q}; t, t') \mathcal{D}^{jl}(\mathbf{Q}; t - t') - \mathcal{C}^{lk}(\mathbf{q}; t', t) \mathcal{D}^{lj}(\mathbf{Q}; t' - t)]. \end{aligned} \quad (9)$$

Only times $t' < t$ contribute due to causality. The function \mathcal{C} is the dynamical spin-spin correlation function,

$$\mathcal{C}^{kl}(\mathbf{q}; t, t') = \frac{1}{N_\Phi} \langle \hat{S}^k(\mathbf{q}, t) \hat{S}^l(-\mathbf{q}, t') \rangle. \quad (10)$$

The spin-density couples in H_{e-ph} to the phonons via

$$\hat{\Phi}(\mathbf{Q}) = \sum_s \left(\lambda_s(-\mathbf{Q}) b_{-\mathbf{Q},s} + \lambda_s(\mathbf{Q})^* b_{\mathbf{Q},s}^\dagger \right); \quad (11)$$

thus, the phonon expectation value \mathcal{D} in (9) is given by

$$\mathcal{D}^{jl}(\mathbf{Q}; t - t') = \langle \hat{\Phi}^j(\mathbf{Q}, t) \hat{\Phi}^l(-\mathbf{Q}, t') \rangle. \quad (12)$$

\mathcal{D} is easily calculated with the equilibrium Hamiltonian H_{ph} of the phonons. ϵ^{ijk} is the antisymmetric tensor. Its origin is the time derivative in (9) which leads to a commutator of $\hat{\mathbf{S}}$ with the spin-density in H_{e-ph} .

It is apparent now that *collective modes* are responsible for the relaxation process. The question is whether one can express the two-particle (four-fermion) correlation function of the interacting system, \mathcal{C} , again by $\mathbf{S}(t)$ and thus derive a closed equation for $\mathbf{S}(t)$. Fortunately, that turns out to be possible in our case: The time dependence of \mathcal{C} is determined by particle-hole excitations (e.g., $\hat{S}^+ = \hat{S}^x + i\hat{S}^y \sim c_{\uparrow}^\dagger c_{\downarrow}$). For H_e , the particle-hole excitations above the ground state are spin-excitons and their dispersion $E_{ex}(q)$ is rigorously known [22,23] at $\nu = 1$,

$$E_{ex}(q) = 2\Delta + \frac{e^2}{\kappa l_B} \sqrt{\frac{\pi}{2}} \left[1 - e^{-\frac{q^2 l_B^2}{4}} I_0\left(\frac{q^2 l_B^2}{4}\right) \right], \quad (13)$$

κ is the dielectric constant, l_B is the magnetic length, and $I_0(x)$ denotes the Bessel function. We now approximate, in the present case of a non-equilibrium state and $\nu \sim 1$, the time dependence of \mathcal{C} with the above dispersion of the spin-exciton, $E_{ex}(q)$. Thus we neglect (i) the difference between the real energy of a single spin-exciton in a non-equilibrium state and $E_{ex}(q)$, and (ii) interactions between the spin-excitations at $\nu \neq 1$. This yields, e.g.,

$$\mathcal{C}^{k+}(\mathbf{q}; t, t') \simeq e^{iE_{ex}(q)(t-t')} \mathcal{C}^{k+}(\mathbf{q}; t, t). \quad (14)$$

The remaining equal-time correlations are then calculated in the Hartree-Fock approximation. This is the weak-scattering approximation in which the effect of the phonons is included in the lowest order in the state, but neglected in the time dependence [17]. Since the characteristic energy for the time dependence is the Zeeman energy Δ , that corresponds to the condition $\Delta \gg 1/\tau$, where the scattering time τ is defined below in Eq. (16). The two approximations above regarding the exciton energy demand that the electronic temperature must be small compared to the Zeeman energy Δ and also, the filling factor must be such that $|\nu - 1| \ll 1$. Now, collecting all terms, we get the resulting kinetic equations for the components of $\mathbf{S}(t)$ ($S^+ = S^x + iS^y$; details of the calculation are deferred to a forthcoming publication):

$$\partial_t S^z(t) = \frac{1}{\tau} \left[1 - \frac{\nu}{2} - S^z(t) \right] \left[\frac{\nu}{2} - S^z(t) \right] - \frac{2}{\tau} \bar{N} S^z(t) \quad (15)$$

$$(\partial_t + i2\Delta) S^+(t) = -\frac{1}{\tau} \left[\frac{1}{2} - S^z(t) + \bar{N} \right] S^+(t).$$

The relaxation time τ and the average phonon number \bar{N} are defined by the two components of the following equation (here, $N(\omega) = [e^{\beta\omega} - 1]^{-1}$)

$$\left[\frac{1/\tau}{\bar{N}/\tau} \right] = \sum_{\substack{\mathbf{Q},s \\ j=x,y}} \frac{\pi}{2} |\lambda_s^j(\mathbf{Q})|^2 \delta(E_{ex}(q) - \omega_{\mathbf{Q},s}) \left[\frac{1}{N(\omega_{\mathbf{Q},s})} \right]. \quad (16)$$

It is quite instructive to formulate again the kinetic equation in terms of the depolarization vector $\boldsymbol{\delta}(t)$. We use $\delta^z(t) = S^z(\infty) - S^z(t)$ and we split the precession term off $S^+(t)$ by redefining $\delta^+(t) = \exp(i2\Delta t) S^+(t)$. Then, the result is

$$\begin{aligned} \tau \partial_t \delta^z(t) &= -\delta^z(t) [\epsilon + \delta^z(t)] \\ \tau \partial_t \delta^+(t) &= -\delta^+(t) \left[\frac{\epsilon}{2} + \delta^z(t) \right]. \end{aligned} \quad (17)$$

Here, the parameter ϵ corresponding to the one used above in the case of the two-state model is defined as $\epsilon = [4\bar{N}(1 + \bar{N}) + (\nu - 1)^2]^{1/2}$. Obviously, Eqs. (17) are quite different from the standard Bloch equations [14].

They are non-linear and it is seen that the relaxation of the transverse component $\delta^+(t)$ depends on the other component. Since the first of the two kinetic equations (17) is identical to (6), the same discussion applies. Observing $\tau \partial_t \ln[\delta^z(t)/\delta^+(t)] = -\epsilon/2$, Eqs. (17) can be explicitly solved with the result

$$\frac{\delta^z(t)}{\delta^z(0)} = e^{-\epsilon t/(2\tau)} \frac{\delta^+(t)}{\delta^+(0)} = \frac{e^{-\epsilon t/\tau}}{1 + \delta^z(0)(1 - e^{-\epsilon t/\tau})/\epsilon}. \quad (18)$$

For any finite value of the parameter ϵ , the leading asymptotic behavior yields a ratio $T_2 = 2T_1$ of the relaxation times. On the other hand, for zero temperature and $\nu = 1$, all normalized components of the depolarization follow *the same function* which is a power law $\propto t^{-1}$.

The relaxation time τ is determined by those phonons, whose energy and in-plane momentum match the energy and the momentum of the spin-excitons, see Eq. (16). Using the model of D'yakonov and Perel' (c.f. Ref. [19,20]), we calculate the electron phonon coupling parameter $\lambda_s^j(\mathbf{Q})$ and perform the summations and integrations in (16) with the following result in the limit of $\Delta = 0$

$$\frac{1}{\tau} = \frac{1}{8} \sqrt{\frac{2}{\pi}} \left(\frac{v}{s}\right)^2 \frac{\hbar}{\rho l_B^5} x_0 e^{-4x_0} W_{1,1}(8x_0). \quad (19)$$

Here, v parametrizes the electron phonon coupling, s is the phonon velocity, ρ the 3-d density, and $W_{1,1}(x)$ the Whittaker function. The parameter x_0 is given by $x_0 = (\hbar s/(\epsilon_c l_B))^2$, where ϵ_c is the Coulomb energy. An estimate with characteristic values for electrons in GaAs at a magnetic field of 10 T yields $\tau \sim 10^{-10} \text{sec}$. This order of magnitude is much smaller than the estimate in Ref. [20] and in agreement with experimental observation [12].

We have studied the spin relaxation in a quantum Hall system for filling factors $\nu \simeq 1$. The resulting kinetic equation (15) is non-linear and quite different from a conventional Bloch equation. This difference is most pronounced for $\nu \rightarrow 1$ and $T \rightarrow 0$; there, a linearization of the kinetic equation fails, since the linear term vanishes at $\nu = 1$, $T = 0$. The relevant time τ (16,19) in the kinetic equation is determined dominantly by the Coulomb interaction between the electrons. —

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